FINAL: ALGEBRA III

Date: **16th Nov 2018**
The total points is **111** and the maximum you can score is **100** points.

A **ring** would mean a **commutative ring with identity**.

(1) (6+6+6+6=24 points) Let k be a field. Let A be the localization of $k[x,y]$ with respect to the multiplicative subset $S = k[x,y] \setminus (x,y)$. Let $B = k[x,y]/(x-y^4)$ and $C = \mathbb{Z}[x]/(2x^2-10)$. Analyze the rings A, B and C to answer the following multiple choice questions about each of them. Write all correct options. No justification needed. No partial credit if a correct option is missing or an incorrect option is written.
 (a) The ring A is:
 (i) Noetherian (ii) PID (iii) UFD (iv) Integral domain
 (b) The ring B is:
 (i) Noetherian (ii) PID (iii) UFD (iv) Integral domain
 (c) The ring C is:
 (i) Noetherian (ii) PID (iii) UFD (iv) Integral domain
 (d) Every nonzero prime ideal of the following ring is also maximal ideal.
 (i) A (ii) B (iii) C (iv) $B \times B$

(2) (8+8+8+8=32 points) Give a counter example to disprove the following statements.
 (a) For a ring R, every torsion module has a nonzero annihilator.
 (b) Every torsion free $\mathbb{Z}$-module is free.
 (c) Every finitely generated module over any ring is a noetherian module.
 (d) For any ring R, M and A two R-modules and N an R-submodule of M, the module $N \otimes_R A$ is isomorphic to an R-submodule of $M \otimes_R A$.

(3) (20 points) Prove or disprove. Let p be a prime number. Then there exist an irreducible polynomial f in $\mathbb{Z}_{(p)}[x]$ such that the ideal (f) is a maximal ideal.

(4) (5+10=15 points) Let R be an integral domain and M an R-module. Define the rank of the R-module M. Show that $\operatorname{rank}(M) = \operatorname{rank}(N) + \operatorname{rank}(M/N)$ where N is a R-submodule of M.

(5) (5+15=20 points) State structure theorem for finitely generated modules over a PID. Let $V_1 = \mathbb{C}[x]/(x^2(x^2-1)(x^3-1))$, $V_2 = \mathbb{C}[y]/(y^2-y)$ and $V = V_1 \oplus V_2$ be $\mathbb{C}$-vector spaces. Let $\phi_1 \in \operatorname{End}(V_1)$, $\phi_2 \in \operatorname{End}(V_2)$ and $\phi \in \operatorname{End}(V)$ be given by $\phi_1(v_1) = \bar{x}v_1$, $\phi_2(v_2) = \bar{y}v_2$ and $\phi(v_1, v_2) = (\bar{x}v_1, \bar{y}v_2)$ $\forall v_1 \in V_1, \forall v_2 \in V_2$ respectively. Find the rational canonical and the Jordan canonical forms of ϕ_1, ϕ_2 and ϕ.